NORMAL DISTRIBUTION AND ECONOMIC APPLICATION PROBLEMS

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Tóm tắt:

Phân phối chuẩn có vị trí quan trọng trong lý thuyết xác suất, có ý nghĩa rất lớn trong thực tế. Rất nhiều biến ngẫu nhiên có luật phân phối chuẩn hoặc gần chuẩn. Những biến ngẫu nhiên có liên quan đến số lượng lớn, chịu ảnh hưởng của các yếu tố cân bằng nhau thường có luật phân phối chuẩn. Bài viết trình bày về phân phối chuẩn cho biến ngẫu nhiên liên tục và những ứng dụng của nó trong lĩnh vực kinh tế.

Từ khóa: *phân phối chuẩn, phân phối chuẩn tắc, biến ngẫu nhiên, xác suất thống kê, ứng dụng trong kinh tế.*

Abstract:

The normal distribution has an important place in probability theory, which has great significance in practice. Many random variables have a normal or near-normal distribution. Random variables that are related to large numbers, influenced by equally balanced factors, often have a normal distribution. The article presents about the normal distribution for continuous random variables and its applications in the field of economics.

Keywords: *normal distribution, standard normal distribution, random variables, statistical probability, applications in economics.*

1. Issue

The normal distribution is an important class of statistical distribution that has a wide range of applications. Many statistical parameters are found to be approximately normally distributed; therefore, the normal distribution is often used for statistical inferences. A variety of natural phenomena either approximately follow a normal distribution or can be transformed to follow a normal distribution. As with any probability distribution, the normal distribution describes how the values of a variable are distributed. It is the most important probability distribution in statistics because it accurately describes the distribution of values for many natural phenomena: people's heights, IQ scores, examination grades, sizes of snowflakes, errors in measurements, lifetimes of lightbulbs, business strategies, weights of loaves of bread, milk production of cows and so on.

So, in this article, we will explore all the concepts about normal distribution in a detailed manner.

2. Normal distribution

Normal distribution, also known as the Gaussian distribution, is a probability distribution that is symmetric about the mean, showing that data near the mean are more frequent in occurrence than data far from the mean. The normal distribution is by far the most important probability distribution. One of the main reasons for that is the Central Limit Theorem (CLT). To give you an idea, the CLT states that if you add a large number of random variables, the distribution of the sum will be approximately normal under certain conditions. The importance of this result comes from the fact that many random variables in real life can be expressed as the sum of a large number of random variables and, by the CLT, we can argue that distribution of the sum should be normal. The CLT is one of the most important results in probability and we will discuss it later on. Here, we will introduce normal random variables. [1]

We first define the standard normal random variable. We will then see that we can obtain other normal random variables by scaling and shifting a standard normal random variable.

A continuous random variable is said to be a standard normal (standard Gaussian) random variable, shown as, if its probability density function (PDF) is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}} ; z \in \mathbb{R}. \quad (1)$$

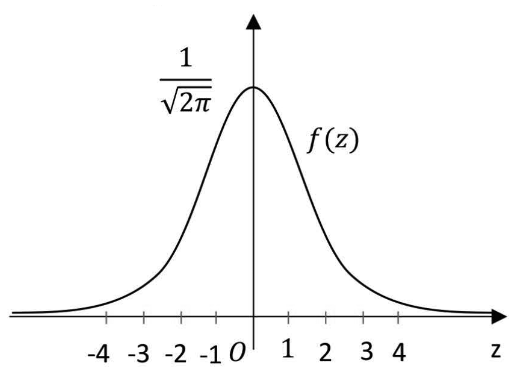

Fig 1. *PDF for standard normal distribution.*

So far, we have shown the following:

If $Z \sim N(0,1)$, then $E(Z) = \mu = 0$.

and $Var(Z) = \sigma^2 = 1$. [2] [3]

Cumulative Distribution Function (CDF) of the standard normal distribution.

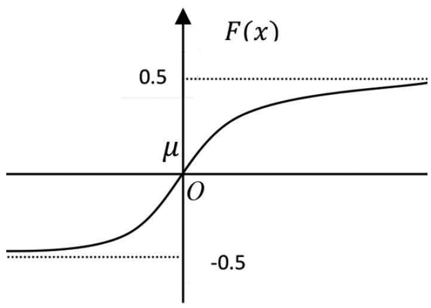

***Fig2.** CDF of the standard normal distribution*

To find the CDF of the standard normal distribution, we need to integrate the PDF function. In particular, we have

$$F(z) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt, \forall z \in \mathbb{R}. \quad (2)$$

This integral does not have a closed form solution. Nevertheless, because of the importance of the normal standard distribution, the values of *F(z)* have been tabulated and many calculators and software packages have this function. We usually denote the standard normal CDF by ϕ

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt, \quad (3)$$

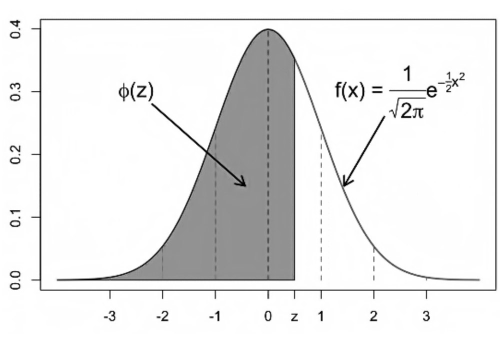

***Fig 3.** The probability density function of the standard normal distribution.*

Since the graph "f(z)" is symmetric about the vertical axis (Fig 1) and $\int_{-\infty}^{+\infty} f(z)dz = 1$, so

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{0} e^{\frac{-t^2}{2}} dt + \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{z} e^{\frac{-t^2}{2}} dt.$$

Set $\varphi(z) = \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{z} e^{\frac{-t^2}{2}} dt$ (4)

Then Φ(z)=0,5+φ(z).

The function φ(z) is called Laplace function. [4] [6]

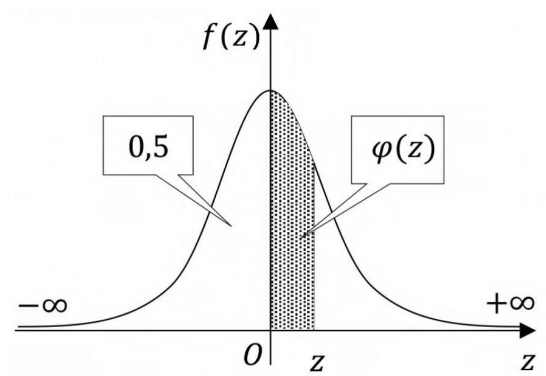

***Fig 4.** The function (Laplace function)*

The function *φ(z)* is an odd function, that is *φ(-z) = - φ(z);*

When z ≥ 4 then *φ(z)* = 0,5. [6]

Normal random variables

Now that we have seen the standard normal random variable, we can obtain any normal random variable by shifting and scaling a standard normal random variable. In particular, define

$X = \sigma Z + \mu$, where $\sigma > 0$.

Then $EX = \sigma EZ + \mu = \mu$,

$Var(X) = \sigma^2 Var(Z) = \sigma^2$. [1] [5]

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We say that X is a normal random variable with mean μ and variance σ^2. We write $X \sim N(\mu; \sigma^2)$.

Conversely, if X~N (μ; σ2) the random variable defined by $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable, i.e., $Z \sim N(0,1)$. To find the CDF of $X \sim N(\mu; \sigma^2)$, we can write:

$$F(x) = P(X \leq x)$$
$$= P(\sigma Z + \mu \leq x) \text{ (Where } Z \sim N(0,1))$$
$$= P\left(Z \leq \frac{X-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{x-\mu}{\sigma}\right).$$

To find the PDF, we can take the derivative of $F(x)$

$$f(x) = \frac{d}{dx}F(x)$$
$$= \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right) \text{ (Chain rule for derivative)}$$
$$= \frac{1}{\sigma}F_Z\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

If Z is a standard normal random variable and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance σ^2, i.e, $X \sim N(\mu; \sigma^2)$ then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}; x \in \mathbb{R}. \quad (5)$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{\frac{-(t-\mu)^2}{2\sigma^2}} dt, \forall x \in \mathbb{R}. \quad (6)$$

$$P(x_1 \leq X \leq x_2) = \varphi\left(\frac{x_2-\mu}{\sigma}\right) - \varphi\left(\frac{x_1-\mu}{\sigma}\right). \quad (7)$$

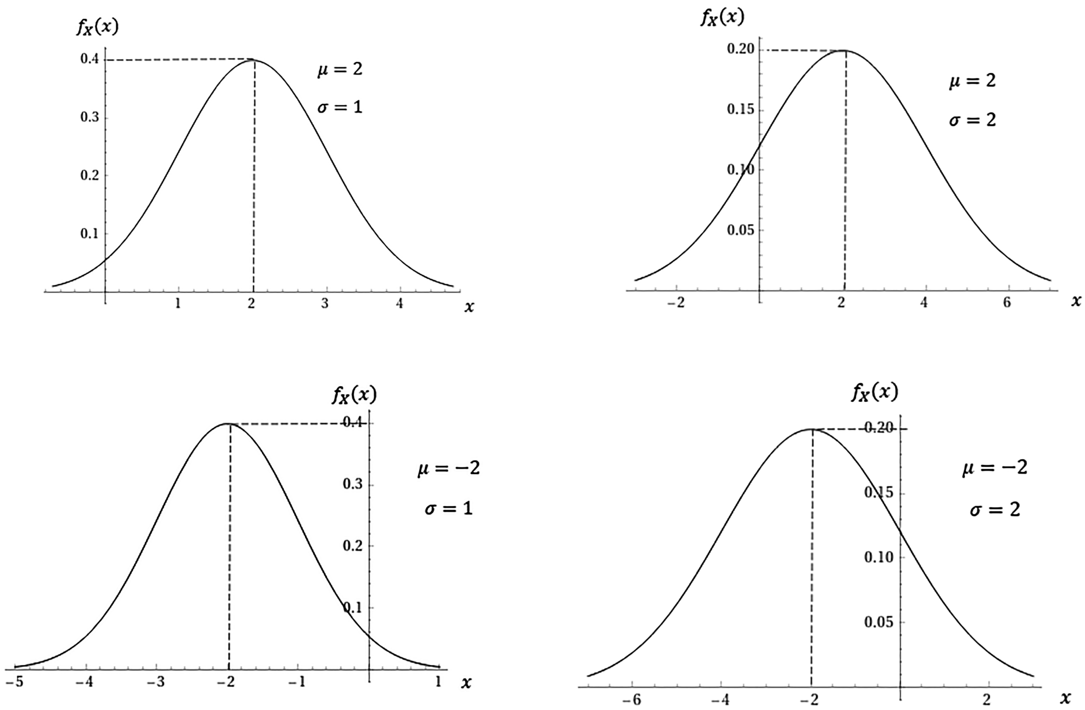

Figure 5. shows the PDF of the normal distribution for several values of μ and σ.

Fig 5. *PDF for normal distributions.*

3. Economics application problem.

In probability, the normal distribution is widely used in many fields to calculate the probability of random variables, especially in the field of economics.

Example 1. A company dealing in item *A* intends to adopt one of two business strategies. Symbols X_1, X_2 (million VND/month) are the profits obtained if the first and second business strategies are applied, respectively.

Knowing that

$X_1 \sim N\left(180;3600\right), X_2 \sim N(220;4900)$ and for the company to survive and develop, the profit earned from the business of item A must reach at least 100 million VND / month.

What plan should company apply to trade in item A? Why?

$$P(X_1 \geq 100) = \varphi\left(\frac{\infty - 180}{60}\right) - \varphi\left(\frac{100-180}{60}\right)$$

$$= \varphi(\infty) - \varphi(-1,33) = \varphi(\infty) + \varphi(1,33)$$

$$= 0,5 + 0,4082 = 0,9082 .$$

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$$P(X_2 \geq 100) = \varphi\left(\frac{\infty - 220}{70}\right) - \varphi\left(\frac{100 - 220}{70}\right)$$

$$= \varphi(\infty) - \varphi(-1,71) = \varphi(\infty) + \varphi(1,71)$$

$$= 0,5 + 0,4564 = 0,9564.$$

$$P(X_2 \geq 100) > P(X_1 \geq 100).$$

The company should choose the second option to trade in item A.

Example 2. The life (years) of an air conditioner follows a normal distribution with a mean of 10 and standard deviation of 6,25. The company needs to provide a warranty period for the above type of air conditioner so that the percentage of air conditioners that need to be warranted during the average use period is 15%. Find the specified period for the warranty of the above type of air conditioner.

Suppose that x_0 is the warranty of the above type of air conditioner.

$$P(X < x_0) = 1 - P(X \geq x_0)$$

$$= 1 - \left[\varphi\left(\frac{\infty - 10}{2,5}\right) - \varphi\left(\frac{x_0 - 10}{2,5}\right)\right]$$

$$= 0,5 + \varphi\left(\frac{x_0 - 10}{2,5}\right).$$

We have $$0,5 + \varphi\left(\frac{x_0 - 10}{2,5}\right) = 0,15.$$

Then $$\varphi\left(\frac{x_0 - 10}{2,5}\right) = -0,35.$$

$$\frac{10 - x_0}{2,5} \approx 1,04.$$

$$x_0 = 7,4.$$

Example 3. The rate of return (%) invested in a project is a normally distributed random quantity. According to the investment committee's assessment, a rate higher than 20% has a probability of 0.1587 and a rate of interest higher than 25% has a probability of 0.0228.

a) Calculate the average interest rate and standard deviation of the project.

b) What is the probability of investing in the project without loss?

a) $P(X > 20) = 0,5 - \varphi\left(\frac{20-\mu}{\sigma}\right) = 0,1587$.

$P(X > 25) = 0,5 - \varphi\left(\frac{25-\mu}{\sigma}\right) = 0,0228$.

$$\begin{cases} \varphi\left(\frac{20-\mu}{\sigma}\right) = 0,3413 \\ \varphi\left(\frac{25-\mu}{\sigma}\right) = 0,0228 \end{cases}$$

$$\begin{cases} \frac{20-\mu}{\sigma} = 1 \\ \frac{25-\mu}{\sigma} = 2 \end{cases}$$

$$\begin{cases} \mu = 15 \\ \sigma = 5 \end{cases}$$

b) $P(X > 0) = 0,5 - \varphi\left(\frac{0-15}{5}\right)$

$= 0,5 + \varphi(3)$

$= 0,5 + 0,4987 = 0,9987$.

4. Summary

The normal distribution is a continuous probability distribution that is symmetrical around its mean, most of the observations cluster around the central peak, and the probabilities for values further away from the mean taper off equally in both directions. It has the remarkable property stated in the so-called central limit theorem. The central limit theorem gives the normal distribution its central place in the theory of sampling since many important problems can be solved by this single patter of sampling variability. As a result, the work on statistical inferences is made simpler. Almost all statistical tests discussed in this text assume normal distributions. Fortunately, these tests work very well even if the distribution is only approximately normally distributed. Some tests work well even with very wide deviations from normality. Finally, if the mean and standard deviation of a normal distribution are known, it is easy to convert back and forth from raw scores to percentiles.

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